Overview

In the chapters presented so far, we have introduced a number of approaches for analyzing electrical circuits, including: Kirchoff’s current law, Kirchoff’s voltage law, circuit reduction techniques, nodal analysis, and mesh analysis. When we have applied the above analysis methods, we have generally assumed that any circuit elements operate linearly. For example, we have used Ohm’s law to model the voltage-current relationships for resistors. Ohm’s law is applicable only for linear resistors – that is, for resistors whose voltage-current relationship is a straight line described by the equation $v = Ri$. Non-linear resistors have been mentioned briefly; in lab assignment 1, for example, we forced a resistor to dissipate an excessive amount of power, thereby causing the resistor to burn out and display nonlinear operating characteristics. All circuit elements will display some degree of non-linearity, at least under extreme operating conditions.

Unfortunately, the analysis of non-linear circuits is considerably more complicated than analysis of linear circuits. Additionally, in subsequent chapters we will introduce a number of analysis methods which are applicable only to linear circuits. The analysis of linear circuits is thus very pervasive – for example, designing linear circuits is much simpler than the design of non-linear circuits. For this reason, many non-linear circuits are assumed to operate linearly for design purposes; non-linear effects are accounted for subsequently during design validation and testing phases.

The concept of treating an electrical circuit as a system has also been introduced previously. In systems-level analysis of circuits, we are primarily interested in the relationship between the system’s input and output signals. Circuits governed by nonlinear equations are considered to be nonlinear systems; circuits whose governing input-output relationship is linear are linear systems. In this chapter, we formally introduce the concept of linear systems. The analysis of linear systems is extremely common, for the reasons mentioned above: structural systems, fluid dynamic systems, and thermal systems are often analyzed as linear systems, even though the underlying processes are often inherently nonlinear. Linear circuits are a special case of linear systems, in which the system consists only of interconnected electrical circuit elements.

Before beginning this chapter, you should be able to:

- State governing equations for the four types of dependent sources (Chapter 1.2.1)
- Represent systems in block diagram form (Chapter 1.7.0)
- Describe systems in terms of input-output equations (Chapter 1.7.0)

After completing this chapter, you should be able to:

- State the defining properties of linear systems
- Determine whether a system is linear

This chapter requires:

- N/A
1.7.1: Linear Systems

Linearity:

Linear systems are described by linear relations between dependent variables. For example, the voltage-current characteristic of a linear resistor is provided by Ohm’s law:

\[ v = R \cdot i \]

where \( v \) is the voltage drop across the resistor, \( i \) is the current through the resistor, and \( R \) is the resistance of the resistor. Thus, the dependent variables – current and voltage – are linearly related. Likewise, the equations we have used to describe dependent sources (provided in chapter 1.2.1):

- Voltage controlled voltage source: \( v_s = \mu v_i \)
- Voltage controlled current source: \( i_s = g v_i \)
- Current controlled voltage source: \( v_s = r i_i \)
- Current controlled current source: \( i_s = \beta i_i \)

all describe linear relationships between the controlled and controlling variables.

All of the above relationships are of the form

\[ y(t) = K x(t) \]  

(1)

where \( x(t) \) and \( y(t) \) are voltages or currents in the above examples. More generally, \( x(t) \) and \( y(t) \) can be considered to be the input and output signals, respectively, of a linear system. Equation (1) is often represented in block diagram form as shown in Figure 1.

![Figure 1. Linear system block diagram.](image)

The output is sometimes called the response of the system to the input. The multiplicative factor \( K \) relating the input and output is often called the system’s gain. Elements which are characterized by relationships of the form of equation (1) are sometimes called linear elements. The equation relating the system’s input and output variables is called the input-output relationship of the system.
Notice that we have allowed the input and output of our system to vary as functions of time. Constant values are special cases of time-varying functions. We will assume that the system gain is not a time-varying quantity.

For our purposes, we will define linearity in somewhat more broad terms than equation (1). Specifically, we will define a system as linear if it satisfies the following requirements:

**Linearity:**

1. If the response of a system to some input \( x_1(t) \) is \( y_1(t) \) then the response of the system to some input \( \alpha x_1(t) \) is \( \alpha y_1(t) \), where \( \alpha \) is some constant. This property is called **homogeneity**.
2. If the response of the same system to an input \( x_2(t) \) is \( y_2(t) \), then the response of the system to an input \( x_1(t)+x_2(t) \) is \( y_1(t)+y_2(t) \). This is called the **additive** property.

The above two properties defining a linear system can be combined into a single statement, as follows: if the response of a system to an input \( x_1(t) \) is \( y_1(t) \) and the system’s response to an input \( x_2(t) \) is \( y_2(t) \), then the response of the system to an input \( \alpha x_1(t)+\beta x_2(t) \) is \( \alpha y_1(t)+\beta y_2(t) \). This property is illustrated by the block diagram of Figure 2. The \( \sum \) symbol in Figure 2 denotes signal summation; the signs on the inputs to the summation block indicate the signs to be applied to the individual signals.

![Figure 2. Block diagram representation of properties defining a linear system.](image)

The above definition of linearity is more general than the expression of equation (1). For example, the processes of differentiation and integration are linear processes according to the above definition. Thus, systems with the input-output relations such as:

\[
y = a\int x dt \quad \text{and} \quad y = b \frac{dx}{dt}
\]

are linear systems. We will use circuit elements which perform integrations and differentiations later when we discuss energy storage elements such as capacitors and inductors.

**Dependent Variables and Linearity:**

Linearity is based on the relationships between dependent variables, such as voltage and current. In order for a system to be linear, relationships between dependent variables must be linear – plots of one dependent variable against another are straight lines. This causes confusion among some students when we begin to talk about time varying signals. Time is not a dependent variable, and plots of voltages or currents as a function of time for a linear system may not to be straight lines.
Although the above definitions of linear systems are fundamental, we will not often use them directly. Kirchoff’s voltage law and Kirchoff’s current law rely upon summing multiples of voltages or currents. As long as the voltage-current relations for individual circuit elements are linear, application of KVL and KCL to the circuit will result in linear equations for the system. Therefore, rather than direct application of the above definitions of linear systems, we will simply claim that an electrical circuit containing only linear circuit elements will be linear and will have linear input-output relationships. All circuits we have analyzed so far have been linear.

**Linearity:**

If all elements in a circuit have linear voltage-current relationships, the overall circuit will be linear.

**Important note about power:**

A circuit’s power is *not a linear property*, even if the voltage-current relations for all circuit elements are linear. Resistors which obey Ohm’s law dissipate power according to $P = iv = \frac{v^2}{R} = i^2 R$. Thus, the power dissipation of a linear resistor is not a linear combination of voltages or currents – the relationship between voltage or current and power is quadratic. Thus, if power is considered directly in the analysis of a linear circuit, the resulting system is **nonlinear**.