Overview

So far, we have considered circuits which have been governed by algebraic relations. These circuits have, in general, contained only power sources and resistive elements. All elements in these circuits, therefore, have either supplied power from external sources or dissipated power. For these resistive circuits, we can apply either time-varying or constant signals to the circuit without really affecting our analysis approach. Ohm's law, for example, is equally applicable to time-varying or constant voltages and currents:

\[ V = I \cdot R \iff v(t) = i(t) \cdot R \]

Since the governing equation is algebraic, it is applicable at every point in time – voltages and currents at a point in time are affected only by voltages and currents at the same point in time.

We will now begin to consider circuit elements which are governed by differential equations. These circuit elements are called dynamic circuit elements or energy storage elements. Physically, these circuit elements store energy which they can later release back to the circuit. The response, at a given time, of circuits which contain these elements is not only related to other circuit parameters at the same time; it may also depend upon the parameters at other times.

This chapter provides a brief overview of what it meant by energy storage in terms of a system-level description of some physical process. Several examples of energy storage elements are presented, for which the reader may have an intuitive understanding. These examples are intended to introduce the basic concepts in a qualitative manner; the mathematical analysis of dynamic systems will be provided in later chapters.

Before beginning this chapter, you should be able to:

- Represent systems in block diagram form (Chapter 1.7.0)
- Write input-output relations for electrical circuits (Chapters 1.7.0, 1.7.3)
- Write the governing equation for a mass-damper system (Chapter 1.7.0)

This chapter requires:

- N/A

After completing this chapter, you should be able to:

- Qualitatively state the effect of energy storage on the type of mathematics governing a system
- Define transient response
- Define steady-state response
Previously, we introduced the concept of representing a physical process as a system. In this viewpoint, the physical process has an input and an output. The input to the system is generated from sources external to the system – we will consider the input to the system to be a known function of time. The output of the system is the system’s response to the input. The input-output equation governing the system provides the relationship between the system’s input and output. A general input-output equation has the form:

\[ y(t) = f\{x(t)\} \]  

(1)

The process is shown in block diagram form in Figure 1.

![Figure 1. Block diagram representation of a system.](image)

The system of Figure 1 transfers the energy in the system input to the system output. This process transforms the input signal \( u(t) \) into the output signal \( y(t) \). In order to perform this energy transfer, the system will, in general, contain elements which both store and dissipate energy. To date, we have analyzed systems which contain only energy dissipation elements. We review these systems briefly below in a systems context. Subsequently, we introduce systems which store energy; our discussion of energy storage elements is mainly qualitative in this chapter and presents systems for which the student should have an intuitive understanding.

**Systems with no energy storage:**

In previous chapters, we considered cases in which the input-output equation is algebraic. This implies that the processes being performed by the system involve only sources and components which dissipate energy. For example, output voltage of the inverting voltage amplifier of Figure 2 is:

\[ V_{OUT} = \left( \frac{R_f}{R_{in}} \right) V_{in} \]  

(2)

This circuit contains only resistors (in the form of \( R_i \) and \( R_{in} \)) and sources (in the form of \( V_{in} \) and the op-amp power supplies) and the equation relating the input and output is algebraic. Note that the op-amp power supplies do not appear in equation (2), since linear operation of the circuit of Figure 2 implies that the output voltage is independent of the op-amp power supplies.

![Figure 2. Inverting voltage amplifier.](image)
One side effect of an algebraic input-output equation is that the output responds instantaneously to any changes in the input. For example, consider the circuit shown in Figure 3. The input voltage is based on the position of a switch; when the switch closes, the input voltage applied to the circuit increases instantaneously from 0V to 2V. Figure 3 indicates that the switch closes at time $t = 5$ seconds; thus, the input voltage as a function of time is as shown in Figure 4(a). For the values of $R_f$ and $R_{in}$ shown in Figure 3, the input-output equation becomes:

$$V_{OUT} = -5V_{in}$$  \hspace{1cm} (3)

and the output voltage as a function of time is as shown in Figure 4(b). The output voltage responds immediately to the change in the input voltage.

![Switched voltage amplifier](image)

**Figure 3.** Switched voltage amplifier.

![Input and output signals](image)

(a) Input voltage  
(b) Output voltage

**Figure 4.** Input and output signals for circuit of Figure 3.
Systems with energy storage:

We now consider systems which contain energy storage elements. The inclusion of energy storage elements results in the input-output equation for the system which is a differential equation. We present the concepts in terms of two examples for which the reader most likely has some expectations based on experience and intuition.

Example 1: Mass-damper system

As an example of a system which includes energy storage elements, consider the mass-damper system shown in Figure 5. The applied force $F(t)$ pushes the mass to the right. The mass’s velocity is $v(t)$. The mass slides on a surface with sliding coefficient of friction $b$, which induces a force which opposes the mass’s motion. We will consider the applied force to be the input to our system and the mass’s velocity to be the output, as shown by the block diagram of Figure 6. This system models, for example, pushing a stalled automobile.

The system of Figure 5 contains both energy storage and energy dissipation elements. Kinetic energy is stored in the form of the velocity of the mass. The sliding coefficient of friction dissipates energy. Thus, the system has a single energy storage element (the mass) and a single energy dissipation element (the sliding friction). In chapter 1.7.0, we determined that the governing equation for the system was the first order differential equation:

$$m \frac{dv(t)}{dt} + bv(t) = F(t) \quad (4)$$

The presence of the energy storage element causes the input-output equation to be a differential equation.
We will examine the effect that the energy storage element has upon the system response in qualitative terms, rather than explicitly solving equation (4). If we increase the force applied to the mass, the mass will accelerate and the velocity of the mass increases. The system, therefore, is converting the energy in the input force to a kinetic energy of the mass. This energy transfer results in a change in the output variable, velocity.

The energy storage elements of the system of Figure 5 do not, however, allow an instantaneous change in velocity to an instantaneous change in force. For example, say that before time t = 0 no force is applied to the mass and the mass is at rest. At time t = 0 we suddenly apply a force to the mass, as shown in Figure 7(a) below. At time t = 0 the mass begins to accelerate but it takes time for the mass to approach its final velocity, as shown in Figure 7(b). This transitory stage, when the system is in transition from one constant operating condition to another is called the transient response. After a time, the energy input from the external force is balanced by the energy dissipated by the sliding friction, and the velocity of the mass remains constant. When the operating conditions are constant, the energy input is exactly balanced by the energy dissipation, and the system’s response is said to be in steady-state. We will discuss these terms in more depth in later chapters when we perform the mathematical analysis of dynamic systems.

![Figure 7(a). Force applied to mass.](image1)

![Figure 7(b). Velocity of mass.](image2)
Example 2: Heating a mass

Our second example of a system which includes energy storage elements is a body which is subjected to some heat input. The overall system is shown in Figure 8. The body being heated has some mass $m$, specific heat $c_p$, and temperature $T_B$. Some heat input $q_{in}$ is applied to the body from an external source, and the body transfers heat $q_{out}$ to its surroundings. The surroundings are at some ambient temperature $T_0$. We will consider the input to our system to be the applied heat input $q_{in}$ and the output to be the temperature of the body $T_B$, as shown in the block diagram of Figure 9. This system is a model, for example, of the process of heating a frying pan on a stove. Heat input is applied by the stove burner and the pan dissipates heat by transferring it to the surroundings.

![Figure 8. Body subjected to heating.](image)

![Figure 9. System block diagram.](image)

The system of Figure 8 contains both energy storage and energy dissipation elements. Energy is stored in the form of the temperature of the mass. Energy is dissipated in the form of heat transferred to the surroundings. Thus, the system has a single energy storage element (the mass) and a single energy dissipation element (the heat dissipation). The governing equation for the system is the first order differential equation:

$$mc_p \frac{dT_B}{dt} - q_{out} = q_{in}$$

The presence of the energy storage element causes the input-output equation to be a differential equation.
We again examine the response of this system to some input in qualitative rather than quantitative terms in order to provide some insight into the overall process before immersing ourselves in the mathematics associated with analyzing the system quantitatively. If the heat input to the system is increased instantaneously (for example, if we suddenly turn up the heat setting on our stove burner) the mass’s temperature will increase. As the mass’s temperature increases, the heat transferred to the ambient surroundings will increase. When the heat input to the mass is exactly balanced by the heat transfer to the surroundings, the mass’s temperature will no longer change and the system will be at a steady-state operating condition. Since the mass provides energy storage, the temperature of the mass will not respond instantaneously to a sudden change in heat input – the temperature will rise relatively slowly to its steady-state operating condition. (We know from experience that changing the burner setting on the stove does not immediately change the temperature of our pan, particularly if the pan is heavy.) The process of changing the body’s temperature from one steady state operating condition to another is the system’s transient response.

The process of changing the body’s temperature by instantaneously increasing the heat input to the body is illustrated in Figure 10. The signal corresponding to the heat input is shown in Figure 10(a), while the resulting temperature response of the body is shown in Figure 10(b).

![Figure 10. Temperature response to instantaneous heat input.](image-url)