2.5.2: Second Order Circuit Natural Response

Overview

In chapter 2.5.1, we determined that the differential equation governing a second-order could be written in the form

\[
\frac{d^2 y(t)}{dt^2} + 2\zeta\omega_n \frac{dy(t)}{dt} + \omega_n^2 y(t) = f(t)
\]

where \(y(t)\) is any system parameter of interest (for example, a voltage or current in an electrical circuit), \(\omega_n\) and \(\zeta\) are the undamped natural frequency and the damping ratio of the system, respectively, and \(f(t)\) is a forcing function applied to the system. In general, \(f(t)\) is an arbitrary function of the physical input to the system. (The physical input to the system can be, for example, a voltage or current source; \(f(t)\) is a function of these power sources. In chapter 2.5.1, we saw examples in which \(f(t)\) was proportional to an applied voltage or current or proportional to the derivative of an applied voltage or current.)

In this chapter, we will develop the homogeneous solution to the above second order differential equation. For the homogeneous case, the forcing function \(f(t) = 0\).

Before beginning this chapter, you should be able to:

- Write differential equations governing second order parallel RLC circuits (Chapter 2.5.1)
- Write differential equations governing second order series RLC circuits (Chapter 2.5.2)
- Determine the natural response of first order electrical circuits (Chapters 2.4.2, 2.4.3)

This chapter requires:

- N/A

After completing this chapter, you should be able to:

-  

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In this chapter, we develop the homogeneous solution to the differential equation provided in equation (12) of chapter 2.5.1. The appropriate differential equation to be solved is thus

\[ \frac{d^2 y_h(t)}{dt^2} + 2\zeta\omega_n \frac{dy_h(t)}{dt} + \omega_n^2 y_h(t) = 0 \]  

(1)

In equation (1), \( y_h(t) \) is the solution to the homogeneous, or unforced differential equation given by equation (1). A second order differential equation requires two initial conditions in order to solve it; we will take our initial conditions to be the value of the function \( y(t) \) at \( t = 0 \) and the derivative of the function \( y(t) \) at \( t = 0 \). We will state our initial conditions as:

\[
\begin{align*}
    y(t = 0) &= y_0 \\
    \frac{dy(t)}{dt} \bigg|_{t=0} &= y'_0
\end{align*}
\]  

(2)

Our approach to the solution of equation (1) will be consistent with our previous approach to the solution of first order homogeneous differential equations: we will assume the form of the differential equation (1) plug this assumed solution into equation (1) and then use our initial conditions to determine any unknown constants in the solution.

Examination of equation (1) leads us to conclude that the solution \( y_h(t) \) of equation (1) must be a function whose form does not change upon differentiation. Thus, we assume (consistent with our approach in chapter 2.4.3) that the solution to equation (1) will be of the form:

\[ y_h(t) = Ke^{st} \]  

(3)

Substituting equation (3) into equation (1) results in

\[ (Ks^2)e^{st} + 2\zeta\omega_n (Ks)e^{st} + K\omega_n^2 e^{st} = 0 \]

The above can be simplified to

\[ \left[ s^2 + 2\zeta\omega_n s + \omega_n^2 \right]Ke^{st} = 0 \]

The solutions to the above equation are \( Ke^{st} = 0 \) and \( s^2 + 2\zeta\omega_n s + \omega_n^2 = 0 \). The first of these results in the trivial solution, \( K = 0 \), which in general will not allow us to satisfy our initial conditions. Thus, in our solution given by equation (3), we choose \( s \) according to:

\[ s^2 + 2\zeta\omega_n s + \omega_n^2 = 0 \]  

(4)

Since equation (4) is quadratic, values of \( s \) which satisfy it are given by:

\[ s = \frac{-2\zeta\omega_n \pm \sqrt{(2\zeta\omega_n)^2 - 4(2\zeta\omega_n)\omega_n^2}}{2} \]
After simplification, this provides:

\[ s = -\zeta\omega_n \pm \omega_n \sqrt{\zeta^2 - 1} \]  

Equations (3) and (5) indicate that there are two possible solutions to equation (1). Since the original differential equation is linear, we know that superposition is valid and our overall solution can be a linear combination of the two solutions provided by equations (3) and (5). Thus, we take our overall solution to be of the form:

\[ y_h(t) = K_1 e^{s_1 t} + K_2 e^{s_2 t} \]

where \( s_1 \) and \( s_2 \) are provided by equation (5) so that

\[ y_h(t) = K_1 e^{(-\zeta\omega_n \pm \omega_n \sqrt{\zeta^2 - 1}) t} + K_2 e^{(-\zeta\omega_n - \omega_n \sqrt{\zeta^2 - 1}) t} \]

which can be re-written as:

\[ y_h(t) = e^{-\zeta\omega_n t} \left[ K_1 e^{\left( \frac{\omega_n}{\sqrt{\zeta^2 - 1}} \right) t} + K_2 e^{\left( -\frac{\omega_n}{\sqrt{\zeta^2 - 1}} \right) t} \right] \]  

The initial conditions, given by equations (2) can be used to determine the unknown constants, \( K_1 \) and \( K_2 \).

Let us briefly examine the form of equation (6) before providing examples of the homogeneous solution for specific circuit-related examples. We do this by examining individual terms in equation (6):

- In equation (6), the term \( e^{-\zeta\omega_n t} \) is an exponential function of the form discussed in chapter 2.1. Thus, we know that this term corresponds to a decaying exponential, as long as the term \( \zeta\omega_n \) is positive.

- There are three possible forms which the term \( e^{\pm \left( \frac{\omega_n}{\sqrt{\zeta^2 - 1}} \right) t} \) can take:
  1. If \( \zeta > 1 \), the terms \( e^{\pm \left( \frac{\omega_n}{\sqrt{\zeta^2 - 1}} \right) t} \) are either growing or decaying exponentials of the form discussed in chapter EA-201 (if \( \zeta > 1 \), \( e^{\left( \frac{\omega_n}{\sqrt{\zeta^2 - 1}} \right) t} \) grows exponentially with time and \( e^{-\left( \frac{\omega_n}{\sqrt{\zeta^2 - 1}} \right) t} \) decays exponentially with time).
  2. If \( \zeta = 1 \), the terms \( e^{\pm \left( \frac{\omega_n}{\sqrt{\zeta^2 - 1}} \right) t} \) are constant and equal to one \( (e^{\pm 0} = 1) \).
  3. If \( \zeta < 1 \), the terms \( e^{\pm \left( \frac{\omega_n}{\sqrt{\zeta^2 - 1}} \right) t} \) are complex exponentials. (The term \( \sqrt{\zeta^2 - 1} = j\sqrt{1 - \zeta^2} \), where \( j = \sqrt{-1} \). Thus, the term \( e^{\pm \left( \frac{\omega_n}{\sqrt{\zeta^2 - 1}} \right) t} = e^{\pm \left( j\frac{\omega_n}{\sqrt{1 - \zeta^2}} \right) t} \) and we have an exponential raised to an imaginary power.)
Before examining the above results in more detail and performing some physical, circuit-related examples, we provide some material (in the following chapter, 2.5.3) relative to complex exponentials and sinusoidal signals. This material will provide us a context within which we can place our solution of equation (6). Chapter 2.5.3 is optional for readers who are comfortable with complex exponential and sinusoidal signals.