Overview

We will now study dynamic systems which are subjected to sinusoidal forcing functions. Previously, in our analysis of dynamic systems, it was necessary to determine both the unforced response (or homogeneous solution) and the forced response (or particular solution) to the given forcing function. In the next several modules, however, we will restrict our attention to only the system’s forced response to a sinusoidal input; this response is commonly called the sinusoidal steady-state system response. This analysis approach is useful if we are concerned primarily with the system’s response after any initial conditions have died out, since we are ignoring any transient effects due to the system’s natural response.

Restricting our attention to the steady-state sinusoidal response allows a considerable simplification in the system analysis: we can solve algebraic equations rather than differential equations. This advantage often more than compensates for the loss of information relative to the systems natural response. For example it is often the case that a sinusoidal input is applied for a very long time relative to the time required for the natural response to die out, so that the overall effects of the initial conditions are negligible.

Steady-state sinusoidal analysis methods are important for several reasons:

- Sinusoidal inputs are an extremely important category of forcing functions. In electrical engineering, for example, sinusoids are the dominant signal in the electrical power industry. The alternating current (or AC) signals used in power transmission are, in fact, so pervasive that any electrical engineers commonly refer to any sinusoidal signal as “AC”. Carrier signals used in communications systems are also sinusoidal in nature.
- The simplification associated with the analysis of steady state sinusoidal analysis is often so desirable that system responses to non-sinusoidal inputs are interpreted in terms of their sinusoidal steady-state response. This approach will be developed when we study Fourier series.
- System design requirements are often specified in terms of the desired steady-state sinusoidal response of the system.

In this module, we introduce the basic concepts relative to sinusoidal steady state analyses before addressing the mathematical details associated with the method. All concepts presented in this module will be revisited in more detail and with full mathematical rigor in later modules.
Before beginning this module, you should be able to:

- Write the equation governing an arbitrary cosine function
- Sketch the sinusoid corresponding to a given cosine function
- Define the natural (or unforced) response of a system in terms of the system inputs and initial conditions (Chapters 2.4, 2.5)
- Define the forced (or particular) response of a system in terms of the system inputs and initial conditions (Chapters 2.4, 2.5)

After completing this module, you should be able to:

- State the relationship between the sinusoidal steady state system response and the forced response of a system
- For sinusoidal steady-state conditions, state the relationship between the input and output frequencies of an LTI system
- State the two parameters used to characterize the sinusoidal steady-state response of an LTI system

This module requires:

- N/A

We will be concerned with sinusoidal signals, which can be written in the form

\[ f(t) = A \cos(\omega t + \theta) \]  \hspace{1cm} (1)

where \( A \) is the amplitude of the sinusoid, \( \omega \) is the angular frequency (in radians/second) of the signal, and \( \theta \) is the phase angle (expressed in radians or degrees) of the signal. \( A \) provides the peak value of the sinusoid, \( \omega \) governs the rate of oscillation of the signal, and \( \theta \) affects the translation of the sinusoid in time. A typical sinusoidal signal is shown in Figure 1.

![Figure 1. Sinusoidal signal.](image-url)
If the sinusoidal signal of Figure 1 is applied to a linear time invariant system, the response of the system will consist of the system’s natural response (due to the initial conditions on the system) superimposed on the system’s forced response (the response due to the forcing function). As we have seen in previous modules, the forced response has the same form as the forcing function. Thus, if the input is a constant value the forced response is constant, as we have seen in the case of the step response of a system. In the case of a sinusoidal input to a system, the forced response will consist of a sinusoid of the same frequency as the input sinusoid. Since the natural response of the system decays with time, the steady state response of a linear time invariant system to a sinusoidal input is a sinusoid, as shown in Figure 2. The amplitude and phase of the output may be different than the input amplitude and phase, but both the input and output signals have the same frequency.

It is common to characterize a system by the ratio of the magnitudes of the input and output signals ($\frac{B}{A}$ in Figure 2) and the difference in phases between the input and output signals ($\phi - \theta$ in Figure 2) at a particular frequency. It is important to note that the ratio of magnitudes and difference in phases is dependent upon the frequency of the applied sinusoidal signal.

![Figure 2. Sinusoidal steady-state input-output relation for a linear time invariant system.](image)

**Example: Series RLC circuit response:**

Consider the series RLC circuit shown in Figure 3 below. The input voltage to the circuit is given by

$$v_s(t) = \begin{cases} 
0, & t < 0 \\
\cos(5t), & t \geq 0 
\end{cases}$$

Thus, the input is zero prior to $t = 0$, and the sinusoidal input is suddenly “switched on” at time $t = 0$. The input forcing function is shown in Figure 3(a). The circuit is “relaxed” before the sinusoidal input is applied, so the circuit initial conditions are:

$$y(0^-) = \left. \frac{dy}{dt} \right|_{t=0^-} = 0$$

![Figure 3. Series RLC circuit; output is voltage across capacitor.](image)
This circuit has been analyzed previously in chapter 2.5.1, and the derivation of the governing differential equation will not be repeated here. The full output response of the circuit is shown in Figure 3(b). The natural response of the circuit is readily apparent in the initial portion of the response but these transients die out quickly, leaving only the sinusoidal steady-state response of the circuit. It is only this steady state response in which we will be interested for the next several modules. With knowledge of the frequency of the signals, we can define both the input and (steady-state) output by their amplitude and phase, and characterize the circuit by the ratio of the output-to-input amplitude and the difference in the phases of the output and input.

![Input signal](image1)

(a) Input signal

![Output signal](image2)

(b) Output signal.

Figure 3. Input and output signals for circuit of Figure 2.