Overview

In this module, the concepts presented in chapters 2.7.0 and 2.7.1 are used to determine the sinusoidal steady-state response of electrical circuits. Sinusoidal steady-state circuit analysis is developed in terms of examples, rather than attempting to develop a generalized approach à priori. The approach is straightforward, so that a general analysis approach can be inferred from the application of the method to several simple circuits.

The overall approach to introducing sinusoidal steady-state analysis techniques used in this module is as follow:

• We first determine the sinusoidal steady-state response of a simple RC circuit, by solving the differential equation governing the system. This results directly in a solution which is a function of time; it is a time domain analysis technique. The approach is algebraically tedious, even for the simple circuit being analyzed.
• We then re-analyze the same RC circuit using complex exponentials and phasors. This approach results in the transformation of the governing time domain differential equation into an algebraic equation which is a function of frequency. It is said to describe the circuit behavior in the frequency domain. The frequency domain equation governing the system is then solved using phasor techniques and the result transformed back to the time domain. This approach tends to be mathematically simpler than the direct solution of the differential equation in the time domain.
• Several other examples of sinusoidal steady-state circuit analysis are then performed using frequency domain techniques in order to demonstrate application of the approach to more complex circuits. It will be seen that, unlike time-domain analysis, the difficulty of the frequency domain analysis does not increase drastically as the circuit being analyzed becomes more complex.

Before beginning this module, you should be able to:

• Write the differential equations governing an electrical circuit (Chapters 2.4.1 to 2.4.5)
• Define the amplitude, frequency, radian frequency, and phase of a sinusoidal signal
• Express sinusoidal signals in phasor form

After completing this module, you should be able to:

• Perform frequency-domain analyses of electrical circuits
• Sketch phasor diagrams of a circuit’s input and output

This module requires:

• N/A
Example 1: RC circuit sinusoidal steady-state response via time-domain analysis

In the circuit of Figure 1, the input voltage is \( u(t) = V_p \cos(\omega t) \) volts and the circuit response (or output) is the capacitor voltage, \( y(t) \). We want to find the steady-state response (as \( t \to \infty \)).

\[ y(t) = \frac{V_p}{RC} \cos(\omega t) \]

Figure 1. RC circuit.

The differential equation governing the circuit is

\[ \frac{dy(t)}{dt} + \frac{1}{RC} y(t) = \frac{V_p}{RC} \cos(\omega t) \]  (1)

Since we are concerned only with the steady-state response, there is no need to determine the homogeneous solution of the differential equation (or, equivalently, the natural response of the system) so we will not be concerned with the initial conditions on the system – their effect will have died out by the time we are interested in the response. Thus, we only need to determine the particular solution of the above differential equation (the forced response of the system). Since the input function is a sinusoid, the forced response must be sinusoidal, so we assume that the forced response \( y_f(t) \) has the form:

\[ y_f(t) = A \cos(\omega t) + B \sin(\omega t) \]  (2)

Substituting equation (2) into equation (3) results in:

\[ -A \omega \sin(\omega t) + B \omega \cos(\omega t) + \frac{1}{RC} \left[ \cos(\omega t) + B \sin(\omega t) \right] = \frac{V_p}{RC} \cos(\omega t) \]  (3)

Equating coefficients on the sine and cosine terms results in two equations in two unknowns:

\[ -A \omega + \frac{B}{RC} = 0 \]
\[ B \omega + \frac{A}{RC} = \frac{V_p}{RC} \]  (4)

Solving equations (4) results in:


\[
A = \frac{V_p}{1 + (\omega RC)^2} \\
B = \frac{V_p \omega RC}{1 + (\omega RC)^2}
\]

Substituting equations (5) into equation (2) and using the trigonometric identity

\[A \cos(\omega t) + B \sin(\omega t) = \sqrt{A^2 + B^2} \cos[\omega t - \tan^{-1}\left(\frac{B}{A}\right)]\]

results in (after some fairly tedious algebra):

\[y_f(t) = \frac{V_p}{\sqrt{1 + (\omega RC)^2}} \cos[\omega t - \tan^{-1}(\omega RC)] \]  

(6)

Note:
In all steps of the above analysis, the functions being used are functions of time. That is, for a particular value of \(\omega\), the functions vary with time. The above analysis is being performed in the time domain.

Example 2: RC circuit sinusoidal steady-state response via frequency-domain analysis

We now repeat Example 1, using phasor-based analysis techniques. The circuit being analyzed is shown in Figure 2(a) below for reference; the input voltage is \(u(t) = V_p \cos(\omega t)\) volts and the circuit response (or output) is the capacitor voltage, \(y(t)\). We still want to find the steady-state response (as \(t \to \infty\)).

\[u(t) = V_p \cos(\omega t)\]

(a) Physical circuit (real-valued input) \hspace{1cm} (b) Conceptual circuit (complex-valued input).

(\text{a}) \hspace{0.5cm} (\text{b})

Figure 2. RC circuit.

In this analysis approach, we replace the physical input, \(u(t) = V_p \cos(\omega t)\), with a conceptual input based on a complex exponential. The complex exponential input is chosen such that the real part of the complex input is equivalent to the physical input applied to the circuit. We now analyze the conceptual circuit with the complex valued input.
The differential equation governing the circuit is the same as in example 1, but with the complex input:

$$\frac{dy(t)}{dt} + \frac{1}{RC} y(t) = V_p e^{j\alpha}$$

(7)

As in example 1, we now assume a form of the forced response. In this case, however, our solution will be assumed to be a complex exponential:

$$y(t) = |Y| e^{j(\alpha + \theta)}$$

(8)

which can be written in phasor form as:

$$y(t) = Ye^{j\alpha}$$

(9)

where the phasor $Y$ is a complex number which can be expressed in either exponential or polar form:

$$Y = |Y| e^{j\theta} = |Y| \angle \theta$$

(10)

Substituting (9) into equation (7) and taking the appropriate derivative results in:

$$j\omega Ye^{j\alpha} + \frac{1}{RC} Ye^{j\alpha} = V_p e^{j\alpha}$$

(11)

we can divide equation (11) by $e^{j\alpha}$ to obtain:

$$j\omega Y + \frac{1}{RC} Y = \frac{V_p}{RC}$$

(12)

Equation (12) can be solved for $Y$:

$$\left( j\omega + \frac{1}{RC} \right) Y = \frac{V_p}{RC} \Rightarrow Y = \frac{V_p}{j\omega + 1/RC}$$

(13)

so that

$$Y = \frac{V_p}{1 + j\omega RC}$$

(14)

The magnitude and phase of the output response can be determined from the phasor $Y$:
2.7.2: Sinusoidal steady-state system response

\[ |Y| = \frac{V_p}{\sqrt{1 + (\omega RC)^2}} \]  

\[ \angle \theta_y = -\tan^{-1}(\omega RC) \]  

(15)

The complex exponential form of the system response is then, from equation (9):

\[ y(t) = \frac{V_p}{\sqrt{1 + (\omega RC)^2}} e^{j(\omega - \tan^{-1}(\omega RC))} \]  

(16)

Since our physical input is the real part of the conceptual input, and since all circuit parameters are real valued, our physical output is the real part of equation (16) and the forced response is:

\[ y_f(t) = \frac{V_p}{\sqrt{1 + (\omega RC)^2}} \cos[\omega t - \tan^{-1}(\omega RC)] \]  

(17)

which agrees with our result from the time-domain analysis of example 1.

Notes:

- The transition from equation (11) to equation (12) removed the time-dependence of our solution. The solution is now no longer a function of time! The solution includes the phasor representations of the input and output, as well as (generally) frequency. Thus, equation (12) is said to be in the phasor domain or, somewhat more commonly, the frequency domain. The analysis remains in the frequency domain until we re-introduce time in equation (17).
- Equations in the frequency domain are algebraic equations rather than differential equations. This is a significant advantage mathematically, especially for higher-order systems.
- Circuit components must have purely real values for the above process to work. We do not prove this, but merely make the claim that the process of taking the real part of the complex exponential form of the system response is not valid if circuit components (or any coefficients in the differential equation governing the system) are complex valued. Fortunately, this is not a strong restriction – complex values do not exist in the physical world.
- The complex exponential we use for our “conceptual” input, \( V_p e^{j\omega t} \), is not physically realizable. That is, we cannot create this signal in the real world. It is a purely mathematical entity which we introduce solely for the purpose of simplifying the analysis. The complex form of the output response given by equation (16) is likewise not physically realizable.
Example 3: Numerical example and Phasor Diagrams

We now examine the circuit shown in Figure 3. This circuit is simply the circuit of Example 2, with \( R = 1 \, \text{k}\Omega, \, C = 1 \, \mu\text{F}, \, V_p = 5 \, \text{V}, \) and \( \omega = 1000 \, \text{rad/second}. \)

![Figure 3. RC circuit.](image)

In phasor form, the input is \( u(t) = U e^{j1000t}, \) so that the phasor \( U \) is \( U = 5e^{j90^\circ} = 5\angle 0^\circ \)

The phasor form of the output is given by equations (15):

\[
|Y| = \frac{V_p}{\sqrt{1 + (\omega RC)^2}} = \frac{5}{\sqrt{1 + (1000 \cdot 1000 \cdot 1 \times 10^{-6})}} = \frac{5}{\sqrt{2}}
\]

\[
\angle \theta_y = -\tan^{-1}(\omega RC) = -\tan^{-1}(1000 \cdot 1000 \cdot 1 \times 10^{-6}) = -\frac{\pi}{4} = -45^\circ
\]

and the phasor \( Y \) can be written as \( Y = \frac{5}{\sqrt{2}} e^{-j45^\circ} = \frac{5}{\sqrt{2}} \angle -45^\circ \)

We can create a phasor diagram of the input phasor \( U \) and the output phasor \( Y \):

![Phasor diagram](image)

The phasor diagram shows the input and output phasors in the complex plane. The magnitudes of the phasors are typically labeled on the diagram, as is the phase difference between the two phasors. Note that since the phase difference between \( Y \) and \( U \) is negative, the output \( y(t) \) lags the input \( u(t) \).
The time-domain form of the output is:

\[ y(t) = \frac{5}{\sqrt{2}} \cos\left(1000t - 45^\circ\right) \]

A time-domain plot of the input and output are shown below. This plot emphasizes that the output lags the input, as indicated by our phasor diagram. The plot below replicates what would be seen from a measurement of the input and output voltages.

**Example 4: RL circuit sinusoidal steady-state response**

In the circuit of Figure 4(a), the input voltage is \( V_p \cos(\omega t + 30^\circ) \) volts and the circuit response (or output) is the inductor current, \( i_L(t) \). We want to find the steady-state response \( i_L(t \to \infty) \).

\[
L \frac{di_L(t)}{dt} + R i_L(t) = u(t)
\]

We apply the conceptual input, \( u(t) = V_p e^{j(\omega t + 30^\circ)} \) as shown in Figure 4(b) to this equation. We can represent this input in phasor form as:
2.7.2: Sinusoidal steady-state system response

\[ u(t) = U e^{j\alpha} \]  
\[ (18) \]

where the phasor \( U = V_p \angle 30^\circ \). Likewise, we represent the output in phasor form:

\[ i_L(t) = I_L e^{j\alpha} \]  
\[ (19) \]

where the phasor \( I_L = |I_L| \angle \theta \).

Substituting our assumed input and output in phasor form into equation (19) results in:

\[ Lj \omega I_L e^{j\alpha} + R I_L e^{j\alpha} = U e^{j\alpha} \]  
\[ (20) \]

As in Example 2, we divide through by \( e^{j\alpha} \) to obtain the frequency domain governing equation:

\[ Lj \omega I_L + RI_L = U \]  
\[ (21) \]

so that

\[ I_L = \frac{U}{R + j\omega L} = \frac{V_p \angle 30^\circ}{R + j\omega L} \]  
\[ (22) \]

so that the phasor \( I_L \) has magnitude and phase:

\[ |I_L| = \frac{V_p}{\sqrt{R^2 + (\omega L)^2}} \]

\[ \theta = 30^\circ - \tan^{-1}\left(\frac{\omega L}{R}\right) \]  
\[ (23) \]

The exponential form of the inductor current is therefore:

\[ i_L(t) = \frac{V_p}{\sqrt{R^2 + (\omega L)^2}} e^{j \left[ \omega t + 30^\circ - \tan^{-1}\left(\frac{\omega L}{R}\right) \right]} \]  
\[ (24) \]

and the actual physical inductor current is

\[ i_L(t) = \frac{V_p}{\sqrt{R^2 + (\omega L)^2}} \cos \left[ \omega t + 30^\circ - \tan^{-1}\left(\frac{\omega L}{R}\right) \right] \]  
\[ (25) \]
Example 5: Series RLC circuit sinusoidal steady-state response

Consider the circuit shown in Figure 5. The input to the circuit is \( v_s(t) = 3\cos(\omega t) \) volts. Find the output \( v(t) \).

![Series RLC circuit](image)

Figure 5. Series RLC circuit.

In chapter 2.5.1, it was determined that the differential equation governing the system is:

\[
\frac{d^2 v(t)}{dt^2} + \frac{R}{L} \frac{dv(t)}{dt} + \frac{1}{LC} v(t) = \frac{1}{LC} v_s(t) \tag{26}
\]

Assuming that the input is a complex exponential whose real part is the given \( v_s(t) \) provides:

\[
v_s(t) = 2e^{j\alpha} \tag{27}\]

The output is assumed to have the phasor form:

\[
v(t) = V e^{j\alpha} \tag{28}\]

where \( V \) contains the (unknown) magnitude and phase of the output voltage. Substituting equations (27) and (28) into equation (26) results in:

\[
-(j\omega)^2 V e^{j\alpha} + \frac{R}{L}(j\omega)V e^{j\alpha} + \frac{1}{LC} V e^{j\alpha} = \frac{1}{LC} 2e^{j\alpha} \tag{29}
\]

Dividing through by \( e^{j\alpha} \) and noting that \( j^2 = -1 \), results in

\[
\left[ \frac{1}{LC} - \omega^2 + j\frac{R}{L} \omega \right] V = \frac{2}{LC}
\]

so that
\[ V = \frac{2/LC}{\frac{1}{LC} - \omega^2 + j\frac{R}{L}\omega} \]  

(30)

The magnitude and phase of \( V \) are

\[ |V| = \frac{2/LC}{\sqrt{\left(\frac{1}{LC} - \omega^2\right)^2 + \left(\frac{R}{L}\omega\right)^2}} \]

\[ \angle V = -\tan^{-1}\left(\frac{\frac{R\omega}{L}}{\frac{1}{LC} - \omega^2}\right) \]

and the capacitor voltage is:

\[ v(t) = \frac{2/LC}{\sqrt{\left(\frac{1}{LC} - \omega^2\right)^2 + \left(\frac{R}{L}\omega\right)^2}} \cos\left(\omega t - \tan^{-1}\left(\frac{\frac{R\omega}{L}}{\frac{1}{LC} - \omega^2}\right)\right) \]  

(31)

The complex arithmetic in this case becomes a bit tedious, but the complexity of the frequency-domain approach is nowhere near that of the time-domain solution of the second-order differential equation.