Overview

In previous chapters, we wrote the differential equation governing the relationship between a circuit’s input and output (the input-output equation) and used this differential equation to determine the response of a circuit to some input. We also characterized the time-domain behavior of the system by examining the circuit’s natural and step responses. We saw that the behavior of a first order circuit can be characterized by its time constant and DC gain, while the response of a second order circuit is characterized by its natural frequency, damping ratio and DC gain. It is important to recognize that these characterizations were independent of specific input parameters; they depended upon the type of response (e.g. a step function or a natural response), but were independent of detailed information such as the amplitude of the step input or the actual values of the initial conditions.

We will now use the steady state sinusoidal response to characterize a circuit’s behavior. As in the case of our time-domain characterization, this characterization will allow the system’s behavior to be defined in terms of its response to sinusoidal inputs, but the characterization will be independent of details such as the input sinusoid’s amplitude or phase angle. (The input sinusoid’s frequency will, however, still be of prime importance.)

Before beginning this module, you should be able to:

- Calculate impedances for resistors, capacitors, and inductors (chapter 2.7.3)
- Analyze frequency domain circuits to determine phasor representations of circuit input and output signals (chapter 2.7.4)
- Draw phasor diagrams of circuit input and output signals (chapter 2.7.3)

After completing this module, you should be able to:

- Define the frequency response of a system
- Define the magnitude response and phase response of a system
- Determine the magnitude and phase responses of a circuit

This module requires:

- N/A

As we have seen, for the case in which a sinusoidal input is applied to a linear system, the system’s forced response consists of a sinusoid with the same frequency as the input sinusoid, but in general having a different amplitude and phase from the input sinusoid. Figure 1 shows the general behavior, in block diagram form. Changes in the amplitude and phase angle between the input and output signals are often used to characterize the circuit’s input-output relationship at the input frequency, \( \omega \). In this chapter, we will demonstrate how this characterization is performed for inputs with discrete frequencies (as in the case of circuits with one or several inputs, each with a single frequency.
component). Later chapters will extend these concepts to the case in which frequency is considered to be a continuous variable.

![Diagram](Image)

Figure 1. Sinusoidal steady-state input-output relation for a linear time invariant system.

In previous chapters (2.7.0 through 2.7.4) we have simplified the analysis of a system’s steady state sinusoidal response significantly. We first represented the sinusoidal signals as complex exponentials in order to facilitate our analysis. We subsequently used phasors to represent our complex exponential signals, as shown in Figure 2; this allowed us to represent and analyze the circuit’s steady state sinusoidal response directly in the frequency domain.

![Diagram](Image)

Figure 2. Phasor representation of sinusoidal inputs and outputs.

In the frequency domain analyses performed to date, we have generally determined the system’s response to a specific input signal with a given frequency, amplitude, and phase angle. We now wish to characterize the system response to an input signal with a given frequency, but an arbitrary amplitude and phase angle. As indicated previously in chapter 2.7.0, we will see that the input-output relationship governing the system reduces to a relationship between the output and input signal amplitudes and the output and input signal phases. The circuit can thus be represented in phasor form as shown in Figure 3. The system’s effect on a sinusoidal input consists of an amplitude gain between the output and input signals \( \left( \frac{B}{A} \right) \) in Figure 3 and a phase difference between the output and input signals \( (\phi - \theta) \) in Figure 3.

![Diagram](Image)

Figure 3. Frequency domain representation of circuit input-output relationship

Rather than perform a rigorous demonstration of this property at this time, we will simply provide a simple example to illustrate the basic concept.
2.7.5: Frequency domain system characterization

Example 1: A sinusoidal voltage, \( v_{in}(t) \), is applied to the circuit to the left below. Determine the frequency-domain relationship between the phasor representing \( v_{in}(t) \) and the phasor representing the output voltage \( v_{out}(t) \).

Since the frequency is unspecified, we leave frequency as an independent variable, \( \omega \), in our analysis. In the frequency domain, therefore, the circuit can be represented as shown to the right above. The frequency domain circuit is a simple voltage divider, so the relation between input and output is:

\[
\frac{V_{out}}{V_{in}} = \frac{1}{R + \frac{1}{j\omega C}}
\]

The factor \( \frac{1}{1 + j\omega RC} \) is a complex number, for given values of \( \omega \), \( R \), and \( C \). It constitutes a multiplicative factor which, when applied to the input, results in the output. This multiplicative factor is often used to characterize the system’s response at some frequency, \( \omega \). We will call this multiplicative factor the frequency response function, and denote it as \( H(j\omega) \). For a particular frequency, \( H(j\omega) \) is a complex number, with some amplitude, \( |H(j\omega)| \), and phase angle, \( \angle H(j\omega) \). For our example, the magnitude and phase of our frequency response function are:

\[
|H(j\omega)| = \frac{1}{\sqrt{1 + (\omega RC)^2}}
\]

\[
\angle H(j\omega) = -\tan^{-1}(\omega RC)
\]

According to the rules of multiplication of complex numbers, when two complex numbers are multiplied, the magnitude of the result is the product of the magnitudes of the individual numbers, and the phase angle of the result is the sum of the individual phase angles. Thus, if the input voltage is represented in phasor form as \( V_{in} = |V_{in}|e^{j\phi} \) and the output voltage is \( V_{out} = |V_{out}|e^{j\phi} \), it is easy to obtain the output voltage from the input voltage and the frequency response function:

\[
|V_{out}| = |V_{in}| \cdot |H(j\omega)|
\]

\[
\angle V_{out} = \angle V_{in} + \angle H(j\omega)
\]
Example 2: Use the frequency response function determined in Example 1 above to determine the response \( v_{\text{out}}(t) \) of the circuit shown below to the following input voltages:

(a) \( v_{\text{in}}(t) = 3\cos(2t+20^\circ) \)
(b) \( v_{\text{in}}(t) = 7\cos(4t-60^\circ) \)

When \( v_{\text{in}}(t) = 3\cos(2t+20^\circ) \), \( \omega = 2 \) rad/sec, \( |V_{\text{in}}| = 3 \) and \( \angle V_{\text{in}} = 20^\circ \). For this value of \( \omega \) and the given values of R and C, the magnitude and phase of the frequency response function are:

\[
|H(j2)| = \frac{1}{\sqrt{1+(\omega RC)^2}} = \frac{1}{\sqrt{1+(2\cdot2\Omega\cdot0.25F)^2}} = \frac{1}{\sqrt{1+4}} = \frac{1}{\sqrt{2}}
\]

\[
\angle H(j2) = -\tan^{-1}(\omega RC) = -\tan^{-1}(2\cdot2\Omega\cdot0.25F) = -\tan^{-1}(1) = -45^\circ
\]

The output amplitude is then the product of \( |V_{\text{in}}| \) and \( |H(j2)| \) and the output phase is the sum of \( \angle V_{\text{in}} \) and \( \angle H(j2) \), so that:

\[
|V_{\text{out}}| = |V_{\text{in}}|\cdot|H(j2)| = 3\cdot\frac{1}{\sqrt{2}} = \frac{3}{\sqrt{2}}
\]

\[
\angle V_{\text{out}} = \angle V_{\text{in}} + \angle H(j2) = 20^\circ + (-45^\circ) = -25^\circ
\]

and the time-domain output voltage is:

\[
v_{\text{out}}(t) = \frac{3}{\sqrt{2}} \cos(2t-25^\circ)
\]

When \( v_{\text{in}}(t) = 7\cos(4t-60^\circ) \), \( \omega = 4 \) rad/sec, \( |V_{\text{in}}| = 7 \) and \( \angle V_{\text{in}} = -60^\circ \). For this value of \( \omega \) and the given values of R and C, the magnitude and phase of the frequency response function are:

\[
|H(j4)| = \frac{1}{\sqrt{1+(\omega RC)^2}} = \frac{1}{\sqrt{5}}, \text{ and } \angle H(j4) = -\tan^{-1}(\omega RC) = -63.4^\circ
\]

The output amplitude is then the product of \( |V_{\text{in}}| \) and \( |H(j4)| \) and the output phase is the sum of \( \angle V_{\text{in}} \) and \( \angle H(j4) \) so that the time-domain output voltage in this case is:

\[
v_{\text{out}}(t) = \frac{7}{\sqrt{5}} \cos(2t-123.4^\circ)
\]
From the above examples we can see that, once the frequency response function is calculated for a circuit as a function of frequency, we can determine the circuit’s steady-state response to any input sinusoid directly from the frequency response function, without re-analyzing the circuit itself.

We conclude this section with one additional example, to illustrate the use of the frequency response function and superposition to determine a circuit’s response to multiple inputs of different frequencies.

**Example 3:** Use the results of examples 1 and 2 above to determine the response $v_{\text{out}}(t)$ of the circuit shown below if the input voltage is $v_{\text{in}}(t) = 3\cos(2t+20^\circ) + 7\cos(4t-60^\circ)$. Plot the input and output waveforms.

Recall, from chapter 2.7.4, that superposition is the only valid approach for performing frequency domain analysis of circuits with inputs at multiple frequencies. Also recall that each frequency can be analyzed separately in the frequency domain, but that the superposition process (the summation of the individual contributions) must be done in the time domain. For this problem, we have contributions at two different frequencies: 2 rad/sec and 4 rad/sec. Luckily, we have determined the individual responses of the circuit to these two inputs in Example 2. Therefore, in the time domain, the two contributions to our output will be:

$$v_1(t) = \frac{3}{\sqrt{2}} \cos(2t - 25^\circ) \quad \text{and} \quad v_2(t) = \frac{7}{\sqrt{5}} \cos(2t - 123.4^\circ).$$

The overall response is then:

$$v_{\text{out}}(t) = v_1(t) + v_2(t) = \frac{3}{\sqrt{2}} \cos(2t - 25^\circ) + \frac{7}{\sqrt{5}} \cos(2t - 123.4^\circ).$$

A plot the input and output waveforms is shown below:
Important points:

- The frequency response function or frequency response describes a circuit’s input-output relationship directly in the frequency domain, as a function of frequency.

- The frequency response is a complex function of frequency $H(j\omega)$ (that is, it is a complex number which depends upon the frequency). This complex function is generally expressed as a magnitude and phase, $|H(j\omega)|$ and $\angle H(j\omega)$, respectively. $|H(j\omega)|$ is called the magnitude response of the circuit, and $\angle H(j\omega)$ is called the phase response of the circuit. The overall idea is illustrated in the block diagram below:

\[
\begin{align*}
\text{Input } U &= A \angle \theta \\
H(j\omega) &= |H(j\omega)| \angle H(j\omega) \\
\text{Output } Y &= B \angle \phi
\end{align*}
\]

- The magnitude response of the circuit is the ratio of the output amplitude to the input amplitude. This is also called the gain of the system. Thus, in the figure above, the output amplitude $B = |H(j\omega)| \cdot A$. Note that the magnitude response or gain of the system is a function of frequency, so that inputs of different frequencies will have different gains.

- The phase response of the circuit is the difference between the output phase angle and the input phase angle. Thus, in the figure above, the output phase $\phi = \angle H(j\omega) + \theta$. Like the gain, the phase response is a function of frequency – inputs at different frequencies will, in general, have different phase shifts.

- Use of the frequency response to perform circuit analyses can be particularly helpful when the input signal contains a number of sinusoidal components at different frequencies. In this case, the response of the circuit to each individual component can be determined in the frequency domain using the frequency response and the resulting contributions summed in the time domain to obtain the overall response.